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Mh4718 Week 11

Week 11

0.0.1 Theoretical Analysis of Fixed Point Iteration

Firstly we recall some results from Real Analysis.

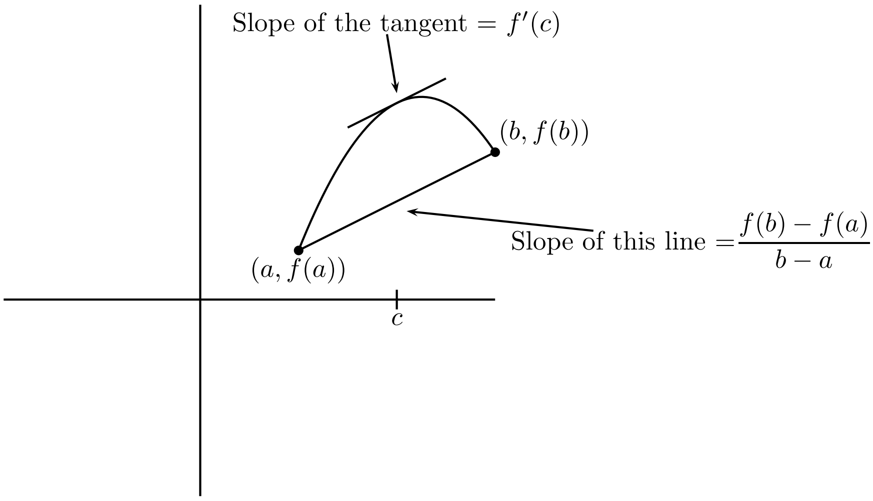
0.0.1.1 The Mean Value Theorem:.

Theorem 0.1 (MVT)

Let f be a real-valued function continuous over $[a, b]$ and differentiable over (a, b). There is a point $c \in (a, b)$ with:

$$f'(c) = \frac{f(a) - f(b)}{b - a}.$$

The figure below illustrates what the Mean Value Theorem is saying. That is, there is some point $c \in (a, b)$ where the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$ (i.e. $f'(c)$) is the same as the slope of the line joining the two points $(a, f(b))$ and $(b, f(b)$.



The Mean Value Theorem is often expressed in the form:

There is some point $c \in (a, b)$ where $f(b) - f(a) = f'(c)(b - a)$.

OR

There is some point $c \in (a, b)$ where $|f(b) - f(a)| = |f'(c)||b - a|$.

In fact if x_1 and x_2 are any two points in the interval $[a, b]$ we have $|f(x_1) - f(x_2)| = |f'(c)||x_1 - x_2|$ where c is some point between x_1 and x_2.

0.0.1.2 The sequence $\{r^n\}$:.
The sequence $\{r^n\}$ converges to 0 when $|r| < 1$.

Now we prove a result which gives us a criterion for the convergence of iteration by F to a fixed point of F.

Theorem 0.2

If F has a fixed point at, say, p and we know that $|F'(x)| < r < 1$ in the interval $(p-\delta, p+\delta)$ for some $\delta > 0$, then iteration with F will converge to p for any initial value $x_0 \in (p-\delta, p+\delta)$

Proof

Remember throughout the following that $F(p) = p$.

Let $x_0, x_1, x_2, \ldots$ be the sequence generated by iteration with F using initial value $x_0 \in (p-\delta, p+\delta)$.
That is, $x_1 = F(x_0), x_2 = F(x_1), x_3 = F(x_2)\ldots.$

First we show that the sequence $x_0, x_1, x_2, \ldots$ stays in the interval $(p-\delta, p+\delta)$.
To do this, just note that for any $x \in (p-\delta, p+\delta)$, using the MVT we have

$$|F(x) - p| = |F(x) - F(p)| = |F'(c)||x-p| < k|x-p| < |x-p| < \delta,$$

That is, if $x \in (p-\delta, p+\delta)$, then $F(x) \in (p-\delta, p+\delta)$.
We can conclude then that the sequence

$$x_0, x_1 = F(x_0), x_2 = F(x_1), \ldots$$

stays inside the interval $(p-\delta, p+\delta)$ if x_0 is in the interval to begin with.
This means that $|F'(x_i)| < r$ for every x_i in the sequence.

Now

$$\begin{aligned} |x_n - p| &= |F(x_{n-1}) - F(p)| = |F'(c)||x_{n-1} - p| < r|x_{n-1} - p| \\ &< r^2|x_{n-2} - p| < \cdots < r^n|x_0 - p| \end{aligned}$$

And since $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$. Then, since $|x_0 - p|$ is fixed, $r^n|x_0 - p|$ becomes arbitrarily small as $n \to \infty$.

$$\Rightarrow \lim_{n\to\infty} x_n = p.$$

□

If we know that F has a fixed point at p and is continuously differentiable then we need only check that $|F'(p)| < 1$ because the continuity of the derivative will guarantee that $|F'(x)| < r < 1$ in some interval around p since, being continuous, it will not suddently “jump” to above 1.
If we then pick x_0 “close enough” to p we can be sure that the sequence

$x_0, x_1, \ldots$ where $x_{n+1} = F(x_n)$ will converge to p.

This requires that we have some idea in advance of the value of p and $F'(p)$ by some other means.

Note also that the rate of convergence is determined by r. That is the smaller $|F'(x)|$ in the interval of iteration then the faster the convergence.

Example 0.3

(i) Let $F(x) = 1 + 0.5\sin(x)$.
We could see from the chart that we drew using Excel that F does have a fixed point p.
$F'(x) = 0.5\sin(x)$ and since $-1 \leq \sin(x) \leq 1$ we see that $-0.5 \leq 0.5\sin(x) \leq 0.5$. That is, $|F'(x)| < 1$ for all $x \in \mathbb{R}$ and so iterating with F using any initial value x_0 will converge to p which incidentally also proves that there is only one fixed point for F

(ii) Let $F(x) = 3 + 2\sin(x)$.
Again we could see from the chart that we drew using Excel that F has a fixed point p between 3 and 4.
$F'(x) = 2\cos(x)$ and a sketch of $F'(x)$ clearly shows that $F'(x) < -1 \Rightarrow |F'(x)| > 1$ in an interval around the fixed point.
The above theorem does not apply therefore to this function and we have no guarantee that the iteration will converge no matter what initial value we choose.